Duality

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The relationship between primal and dual problems

- Duality is the formulation of one problem in two different ways
- As a matter of convention, the original problem is called **the primal problem**
- Associated to any LP there is another LP, called **the dual problem**, whose formulation is different but whose solution gives "identical results" to the original, primal, problem.

The primal problem

$$\text{Max } z = 3x_1 + 5x_2 \qquad (0)$$

$$\begin{cases} x_1 & & \leq & 4 & (1) \\ & 2x_2 & \leq & 12 & (2) \\ 3x_1 & + \; 2x_2 & \leq & 18 & (3) \\ x_1, & x_2 & \geq & 0 & (4) \end{cases}$$

Primal problem

Some notation

x_1, x_2 - productive activities
RHS 4, 12, 18 - resources
(1), (2), (3) - structural constraints
(4) nonnegative constraints
(0) the objective is to Max profit

How to find the dual of a Max problem in which all variables are ≥ 0 and all constraints are of $\leq$ (standard form)?

The relationship between primal and dual problems

- The number of variables of the dual will equal the number of structural constraints in the primal problem $\longrightarrow y_1, y_2, y_3$.
- The number of structural constraints in the dual will equal the number of variables in the primal problem.
- The objective function coefficients for the dual are the RHS in primal problem.
  dual "obj function" $4y_1 + 12y_2 + 18y_3$
- The RHS values for the dual problem correspond to the obj function coefficients of the primal problem.
- If all variables in the primal Max problem are ≥ 0 and all constraints are $\leq$ then the dual constraints are are ≥ 0 and all variables are ≥ 0 $(y_1, y_2, y_3 \geq 0)$.

The primal problem

$$\text{Max } z = 3x_1 + 5x_2$$
$$\begin{cases} x_1 & & & \leq & 4 & \longmapsto y_1 \\ & & 2x_2 & \leq & 12 & \longmapsto y_2 \\ 3x_1 & + & 2x_2 & \leq & 18 & \longmapsto y_3 \\ x_1, & & x_2 & \geq & 0 & \end{cases}$$

Primal problem

$$\text{min } w = 4y_1 + 12y_2 + 18y_3$$
$$\begin{cases} y_1 & & +3y_3 & \geq & 3 \\ & 2y_2 & +2y_3 & \geq & 5 \\ y_1, & y_2 & y_3 \geq & & 0 \end{cases}$$

Dual problem

Constraints of the dual problem are build by reading down the primal problem. The dual constraint i corresponds to the primal variable x_i.

Writing the dual problem of the Poet's problem

Max $z = 90x_1 + 120x_2$ (\$/y)
$x_1 \leq 40$ (ha of red pine)
$x_2 \leq 50$ (ha of hardwoods)
$2x_1 + 3x_2 \leq 180$ (days of work/y)
$x_1, x_2 \geq 0$

Primal problem

min $w = 40y_1 + 50y_2 + 180y_3$
$y_1 + 2y_3 \geq 90$
$y_2 + 3y_3 \geq 120$
$y_1, y_2, y_3 \geq 0$

Dual problem

Primal-Dual relationships

- Weak duality - if x is a primal feasible solution, for a Max LP problem, and y is a dual feasible solution then $z = cx \leq w = yb$.
  Example: Let x be a primal feasible solution with $z = 110$. Then, weak duality implies that any dual feasible solution will have $w \geq 110$.
- Strong duality - if x^* is a primal feasible solution and y^* is a dual feasible solution then $cx^* = y^* b$.
- Symmetry - The dual of the dual problem is the primal problem.

Primal-Dual relationships

- Duality theorem
  - If one problem has **feasible solutions and a bounded objective function value** then so does the other problem.
  - If one problem has **feasible solutions and an unbounded objective function value** then the other problem has no feasible solution.
  - If one problem has **no feasible solutions** then the other problem has either no feasible solutions or an unbounded objective function.

Duality and shadow prices

- The shadow price for resource i, y_i^* measures the marginal value of this resource. That is, the shadow price of the *ith* constraint is the amount by which the optimal z-value is improved if we increase b_i by 1 unit (as long as the current solution remains optimal).
- The shadow price of the *ith* constraint of a max problem is the optimal value of the *ith* dual variable.

Duality and shadow prices for the Poet's problem

- The dual optimal solution is $y_1^* = 10$, $y_2^* = 0$, $y_3^* = 40$.
- These shadow prices show that:
  - One additional hectare of red pine land would increase the poets annual revenues by $10.
  - Extra hardwoods would be worth nothing. This is consistent with the fact that in the best primal solution we found that about 16.7 ha of hardwoods were not used.
  - The third shadow price shows that one additional day working in the woods is worth $40 to the poet.

Bibliography

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